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Calculation of inviscid three-dimensional supersonic flows with heat and mass addition

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The method follows previous work in that the streamlines are constructed simultaneously with the solution of the gas-dynamic equations written in streamwise coordinates. The flow envisaged is three dimensional, which implies that two unknown parameters need to be chosen in order to guide each free streamline over an incremental distance. In general this leads to two errors in cross-plane pressure gradients which are reduced to zero by Newton's method, and the calculation then proceeds to the next downstream step and so on until the required flow field is complete. Heat and/or mass addition is allowed if required and the ratio of specific heats is permitted to vary with temperature.

Keywords: supersonic; 3D flows; additives; flowfield; gas dynamics; streamlines

1. Introduction

A method of calculation has been described for supersonic two-dimensional inviscid flow subject to a known distribution of heat addition (Broadbent 1991). In this method, the calculation marches downstream along the streamlines, which are, of course, initially unknown. The procedure, therefore, is to make an initial guess for the streamline development over a very short distance and then the governing continuity and momentum equations in streamwise coordinates can be readily solved to give a local distribution of pressure, density and flow speed. These interim results are then substituted into the energy equation to give a distribution of heat addition, which will in general, however, differ from that prescribed thus forming an error distribution. Adjustment of the streamline slopes assumed in the initial guess then leads to a change in the error distribution and Newton's method is used to reduce this to zero.

In the present paper a somewhat similar method is described for three-dimensional flow, suitable, for example, for a hypersonic intake in which heating (or cooling) may be used. In this case, however, the elementary method of solution outlined above no longer applies, because each free streamline requires two unknowns to fix its change of path and hence there will in general be two errors in the gas-dynamic properties of the associated streamtube flow. The procedure, therefore, is that each streamline is allotted a streamtube of infinitesimal cross-section and the gas-dynamic properties follow from a quasi-one-dimensional analysis along the streamtube. This leads to a pressure distribution in the cross-plane that will in general not match the cross-plane pressure gradients derived from the local streamline curvature. Since there are two components of the cross-plane pressure gradient to be matched for each streamline and two unknowns fixing the path of each streamline, a consistent set of equations can be derived which are satisfied by Newton's method as in the earlier work.

The method is described in more detail in § 2 together with details of the streamline geometry and the associated (infinitesimal) streamtube area. The gas dynamics is discussed in § 3 and includes allowance for known mass addition as well as known heat addition. With regard to the gas dynamics, the only variation from a perfect gas that is allowed is that the ratio of specific heats, γ, is permitted to vary with temperature, and thermal equilibrium is everywhere assumed to hold. In some applications for air the ad hoc formula

$$\gamma(\bar{T}) = 1.4 - 0.12(1 - \exp(-(\bar{T}/1200)^2)), \tag{1.1}$$

has been used; it gives a reasonably close approximation over a reasonable working range. Here $\bar{T}$ is the absolute temperature in kelvins. Where more complicated chemistry is involved, a method such as that of Clarke (1991) is needed.

2. General description and streamline geometry

We assume a wholly supersonic flow without shocks along a duct which for simplicity has a quadrilateral cross-section starting from a rectangular upstream inlet. At the inlet cross-section the flow is in the x-direction and the sides of the rectangle are parallel to the y- and z-directions, respectively. Since the application is to hypersonic flight, the flow is assumed to remain broadly in the x-direction and changes in cross-section and flow direction are gradual. The method can be adapted to cover other cross-sections, or the presence of shocks, but for clarity the simple assumptions just outlined are assumed throughout this paper.

The flow at any cross-section is given by that along an array of streamtubes, each of infinitesimal cross-sectional area, that cut the cross-section at a set of points to be determined by the calculation together with the corresponding streamline direction and curvature. The cross-sectional area of each streamtube is allowed to vary along the streamtube as the local flow expands or contracts. At the inlet it is convenient to choose a rectangular array of streamlines, with N_y in the y-direction and N_z in the z-direction making N_s in all, where $N_\mathrm{s} = N_y N_z$ (figure 1). Further assumptions are then that there is no separation from the walls or corners and that streamlines do not become entangled.

The method used follows the development of the flow from a plane $x = x_1$, say, where conditions are known, to an adjacent downstream plane $x = x_2$; all lengths are non-dimensional with respect to a chosen length $\bar{d}_0$, say.

Let

$$\xi = \frac{x - x_1}{x_2 - x_1}, \qquad x = (x_2 - x_1)\xi + x_1, \tag{2.1}$$

so that ξ runs from 0 to 1 across the gap. In this gap it is assumed that the shape of a streamline can be represented by cubics in ξ, thus

$$\left.\begin{aligned} y_\mathrm{s} &= a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \equiv A(\xi), \\ z_\mathrm{s} &= b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 \equiv B(\xi). \end{aligned}\right\} \tag{2.2}$$

This representation can be expected to become increasingly accurate as the gap $(x_2 - x_1)$ is reduced. Of the eight coefficients in (2.2), six are determined at the

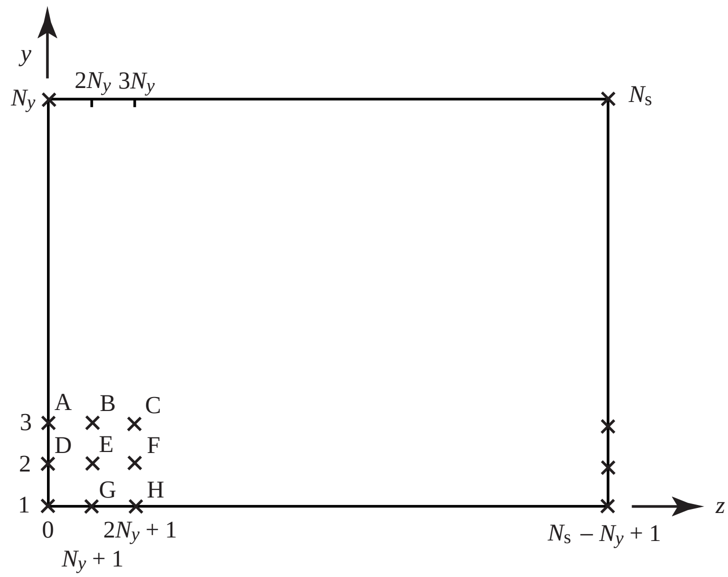

Figure 1. Cross-section at an inlet in the plane $x = 0$, showing the streamline numbering and part of the streamline array. At later sections, $x > 0$, the geometry of the duct and the array will in general be different and non-rectangular but topologically equivalent.

plane x_1 by continuity of position, direction and curvature (since a sudden change in curvature would imply a sudden change in normal pressure gradient, which could only occur at a shock) leaving two unknowns per streamline. The no-separation assumption, however, constrains those streamlines running along the walls to leave only one unknown per streamline, and for the four corner streamlines there are no unknowns since their path is fully determined by the known shape of the walls. The total number of unknowns is thus $2(NX-2)(NY-2)+2(NX-2)+2(NY-2)$, or $2(NS-NX-NY)$. Another way of looking at this is to note that each free streamline needs two free parameters to adjust its intercept with the plane $x = x_2$, each wall streamline needs one such parameter and each corner streamline needs none, leading again to $2(NS-NX-NY)$ unknowns.

To solve for these unknowns, we first assume trial values which determine the path of the streamline through the gap $x_2 - x_1$, from which the change in cross-sectional area can be deduced. Since the streamtubes are infinitesimally thin, quasi-one-dimensional equations of gas dynamics can be written to give formulae for $\partial m_a/\partial s, \partial u/\partial s$ and $\partial p/\partial s$ in terms of the cross-sectional area A_s of the streamtube and the variables themselves, m_a, u and p, where m_a, u and p are non-dimensional forms of the mass flux, the flow speed and the pressure respectively along the streamline; s is the corresponding streamwise coordinate. These equations can be recast as gradients with respect to x, which is more convenient than s, and then integrated by, for example, fourth-order Runge–Kutta to give a pressure distribution over the plane x_2. In particular, the pressure gradients in the plane x_2 can be calculated, with two components for the free streamlines and one component for each wall streamline away from the corners. But the same pressure gradients can also be calculated from the known curvature of the streamlines as they cross the plane x_2, and the fact that the two sets must be equal yields the necessary $2(NS-NX-NY)$ equations. In general, the equations will not be satisfied by the trial values of the unknowns and Newton's method is used to converge onto a valid solution. The whole process is then repeated between the plane $x = x_2$ and the next adjacent downstream plane $x = x_3$, and so on until the solution is complete for the length of the duct.

To illustrate the choice of streamtube area for each streamtube, it is convenient to refer to the inlet sketch in figure 1. As the flow develops from this plane $x = 0$ to some downstream plane $x = x_1$, intercepts labelled $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{G}$ and H on figure 1 will assume a corresponding though distorted array at this new plane. A non-dimensional area A_x on the plane $x = x_1$ is then assumed to be given by: for a corner point such as O,

$$A_x = \frac{(\mathrm{ODG})_{x=x_1}}{(\mathrm{ODG})_{x=0}}; \tag{2.3}$$

for any other edge point such as D,

$$A_x = \frac{(\mathrm{ODE}+\mathrm{DAE})_{x=x_1}}{(\mathrm{ODE}+\mathrm{DAE})_{x=0}}; \tag{2.4}$$

for a midstream point such as E,

$$A_x = \frac{(\mathrm{DBE}+\mathrm{EBF}+\mathrm{GEF}+\mathrm{GDE})_{x=x_1}}{(\mathrm{DBE}+\mathrm{EBF}+\mathrm{GEF}+\mathrm{GDE})_{x=0}}. \tag{2.5}$$

Here $(\mathrm{ODE}+\mathrm{DAE})_{x=x_1}$ is the sum of the area of the triangle in the plane $x = x_1$ formed by the streamlines starting at O, D and E in the plane $x = 0$ and the area of the corresponding streamlines starting from D, A and E; with equivalent notation throughout (2.3)–(2.5). The true cross-sectional area of the streamtube A_s will differ from A_x unless the streamline happens to be normal to the plane $x = \text{const.}$, and is given by

$$A_\mathrm{s} = A_x t_x \tag{2.6}$$

where t_x is the x-component of the tangent vector $\boldsymbol{t}$ to the streamline at the plane ($t_x = 1$ at $x = 0$).

The tangent vector $\boldsymbol{t}(\xi)$ and the related geometric properties of the principal normal $\boldsymbol{n}(\xi)$ and the radius of curvature $R(\xi)$ follow from the position vector of the streamline $\boldsymbol{r}(\xi)$. There follows, with all differentials being for a constant streamline,

$$\boldsymbol{r}(\xi) = ((x_2 - x_1)\xi + x_1)\boldsymbol{i} + A(\xi)\boldsymbol{j} + B(\xi)\boldsymbol{k}, \tag{2.7}$$

where $A(\xi), B(\xi)$ are given by (2.2).

$$\boldsymbol{t}(\xi) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{r}/\mathrm{d}\xi}{\mathrm{d}s/\mathrm{d}\xi} = \frac{\mathrm{d}\boldsymbol{r}/\mathrm{d}\xi}{|\mathrm{d}\boldsymbol{r}/\mathrm{d}\xi|} = \frac{\boldsymbol{r}'(\xi)}{D^{1/2}}, \tag{2.8}$$

where

$$\boldsymbol{r}'(\xi) = (x_2 - x_1)\boldsymbol{i} + A'(\xi)\boldsymbol{j} + B'(\xi)\boldsymbol{k} \tag{2.9}$$

from (2.7) and

$$D = (x_2 - x_1)^2 + A'^2(\xi) + B'^2(\xi), \tag{2.10}$$

giving

$$\boldsymbol{t}(\xi) = ((x_2 - x_1)\boldsymbol{i} + A'(\xi)\boldsymbol{j} + B'(\xi)\boldsymbol{k})D^{-1/2}. \tag{2.11}$$

Further,

$$\frac{\boldsymbol{n}}{R} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{t}/\mathrm{d}\xi}{\mathrm{d}s/\mathrm{d}\xi} = \frac{\boldsymbol{t}'(\xi)}{D^{1/2}}, \tag{2.12}$$

i.e.

$$\begin{aligned}\frac{\boldsymbol{n}}{R} &= D^{-1}(A''(\xi)\boldsymbol{j} + B''(\xi)\boldsymbol{k}) \\ &\quad - D^{-2}((x_2 - x_1)\boldsymbol{i} + A'(\xi)\boldsymbol{j} + B'(\xi)\boldsymbol{k})(A'(\xi)A''(\xi) + B'(\xi)B''(\xi)).\end{aligned} \tag{2.13}$$

Also, since $|\boldsymbol{n}| = 1$, by definition,

$$\frac{1}{R} = \left|\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s}\right|, \tag{2.14}$$

$$\boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{t}/\mathrm{d}s}{|\mathrm{d}\boldsymbol{t}/\mathrm{d}s|} = R\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \frac{\boldsymbol{t}'(\xi)}{|\boldsymbol{t}'(\xi)|}, \tag{2.15}$$

where alternative forms are given in (2.12) and (2.15) for convenience in different parts of a computer program.

3. Gas-dynamic relations

It is convenient to express the physical quantities non-dimensionally. Pressure p and density ρ are given in terms of their values at the inlet:

$$\bar{p} = \bar{p}_0 p, \qquad \bar{\rho} = \bar{\rho}_0 \rho \tag{3.1}$$

where a bar denotes a dimensional quantity and suffix 0 denotes inlet conditions. Speeds are non-dimensionalized by $(\bar{p}_0/\bar{\rho}_0)^{1/2}$, e.g. for the flow speed u,

$$\bar{u} = u(\bar{p}_0/\bar{\rho}_0)^{1/2}. \tag{3.2}$$

Allowance is made for the addition of mass (e.g. fuel mass) at a rate $\bar{m}_\mathrm{f}$ per unit volume per second, and heat $\bar{Q}$ per unit volume per second, with

$$\bar{d}_0\bar{m}_\mathrm{f} = (\bar{p}_0/\bar{\rho}_0)^{1/2} m_\mathrm{f}, \qquad \bar{d}_0\bar{\rho}_0^{1/2}\bar{Q} = \bar{p}_0^{3/2} Q. \tag{3.3}$$

Non-dimensional temperatures are defined in terms of the inlet temperature $\bar{T}_0$ in kelvins:

$$T = \bar{T}_0 T, \qquad \bar{T}_\mathrm{f} = \bar{T}_0 T_\mathrm{f}, \qquad \bar{C}v_\mathrm{f} = \bar{C}v_0 Cv_\mathrm{f} \tag{3.4}$$

where the subscript 'f' refers to the added mass, Cv_0 is the specific heat at constant volume for the inlet flow and $\bar{C}v_\mathrm{f}$ is that of the added mass. The equations used assume perfect mixing of the added mass, that the gas constant remains effectively constant although the ratio of specific heats can vary with temperature, and that thermal equilibrium is maintained. Thus

$$\gamma = \gamma(T), \qquad T = p/\rho = pu/m_\mathrm{a}, \tag{3.5}$$

where $m_{\mathrm{a}} = \rho u$ is the local mass flux per unit area. The following equations of motion may now be written (the continuity equation in (3.6), the three components of the momentum equations in (3.7) for the direction $\boldsymbol{t}$, (3.8) for the direction $\boldsymbol{n}$ and (3.9) for the orthogonal direction $\boldsymbol{b}$, and the energy equation (3.10)):

$$\frac{\partial}{\partial s}(m_{\mathrm{a}} A_{\mathrm{s}}) = m_{\mathrm{f}} A_{\mathrm{s}}, \tag{3.6}$$

$$m_{\mathrm{f}} u + m_{\mathrm{a}} \frac{\partial u}{\partial s} = -\frac{\partial p}{\partial s}, \tag{3.7}$$

$$\frac{m_{\mathrm{a}} u}{R} = -\frac{\partial p}{\partial n}, \tag{3.8}$$

$$\frac{\partial p}{\partial b} = 0, \tag{3.9}$$

$$\frac{\partial}{\partial s}\left(\frac{\gamma}{\gamma - 1} p u A_{\mathrm{s}}\right) + \frac{1}{2}\frac{\partial}{\partial s}(m_{\mathrm{a}} u^2 A_{\mathrm{s}}) = m_{\mathrm{f}} \frac{A_{\mathrm{s}} C v_{\mathrm{f}} T_{\mathrm{f}}}{\gamma_0 - 1} + Q A_{\mathrm{s}}. \tag{3.10}$$

Some manipulation leads to formulae for the streamwise gradients:

$$\frac{\partial m_{\mathrm{a}}}{\partial s} = m_{\mathrm{f}} - \frac{m_{\mathrm{a}}}{A_{\mathrm{s}}}\frac{\partial A_{\mathrm{s}}}{\partial s}, \tag{3.11}$$

$$\frac{\partial u}{\partial s} = \frac{(e_3 + e_2 m_{\mathrm{f}} u)}{(e_1 - m_{\mathrm{a}} e_2)}, \tag{3.12}$$

$$\frac{\partial p}{\partial s} = -\frac{(e_1 m_{\mathrm{f}} u + e_3 m_{\mathrm{a}})}{(e_1 - m_{\mathrm{a}} e_2)}, \tag{3.13}$$

where

$$\left.\begin{aligned}
e_1 &= \frac{1}{\gamma - 1} + M^2 - \frac{p u \gamma'(T)}{\gamma m_{\mathrm{a}} (\gamma - 1)^2}, \\
e_2 &= \frac{u}{(\gamma - 1) p} - \frac{u^2 \gamma'(T)}{\gamma m_{\mathrm{a}} (\gamma - 1)^2}, \\
e_3 &= \frac{m_{\mathrm{f}} C v_{\mathrm{f}} T_{\mathrm{f}}}{\gamma p (\gamma_0 - 1)} + \frac{Q}{\gamma p} - \frac{u}{(\gamma - 1) A_{\mathrm{s}}}\frac{\partial A_{\mathrm{s}}}{\partial s} \\
&\quad - \frac{m_{\mathrm{f}} u^2}{2 \gamma p} + \frac{u \gamma'(T)}{\gamma (\gamma - 1)}\left(\frac{p u}{m_{\mathrm{a}}}\right)\left(\frac{1}{A_{\mathrm{s}}}\frac{\partial A_{\mathrm{s}}}{\partial s} - \frac{m_{\mathrm{f}}}{m_{\mathrm{a}}}\right),
\end{aligned}\right\} \tag{3.14}$$

and M is the local Mach number, i.e. $M = u/(\gamma p/\rho)^{1/2}$. Along a streamline, however, $\mathrm{d}s = \mathrm{d}x/t_x$ so that equations (3.11)–(3.13) may be written

$$\left(\frac{\mathrm{d} m_{\mathrm{a}}}{\mathrm{d}x}\right)_{\mathrm{s}} = \frac{m_{\mathrm{f}}}{t_x} - \frac{m_{\mathrm{a}}}{A_{\mathrm{s}}}\left(\frac{\mathrm{d} A_{\mathrm{s}}}{\mathrm{d}x}\right)_{\mathrm{s}}, \tag{3.15}$$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{\mathrm{s}} = \frac{e_3 + e_2 m_{\mathrm{f}} u}{t_x (e_1 - m_{\mathrm{a}} e_2)}, \tag{3.16}$$

$$\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{\mathrm{s}} = -\frac{(e_1 m_{\mathrm{f}} u + e_3 m_{\mathrm{a}})}{t_x (e_1 - m_{\mathrm{a}} e_2)}, \tag{3.17}$$

where the suffix 's' denotes that a streamline is being followed, where e_1, e_2 and e_3 are given by (3.14) and in e_3

$$\frac{1}{A_\mathrm{s}}\frac{\partial A_\mathrm{s}}{\partial s} \equiv \frac{t_x}{A_\mathrm{s}}\left(\frac{\mathrm{d}A_\mathrm{s}}{\mathrm{d}x}\right)_\mathrm{s}. \tag{3.18}$$

The equations (3.15)–(3.17) with (3.14) and (3.18) can now be integrated by the fourth-order Runge–Kutta procedure over the range $x = x_1$ to x_2 as long as the added quantities m_f and Q are known. To do this requires the streamtube area to be known as a function of x. Equations (2.3)–(2.5) show how A_x can be calculated at any plane $x =$ constant in terms of triangles with apexes given by (2.2) in terms of ξ. Since the area of a triangle, A_t, say, is given by

$$A_\mathrm{t} = ((s_\mathrm{t} - \ell_1)(s_\mathrm{t} - \ell_2)(s_\mathrm{t} - \ell_3)s_\mathrm{t})^{1/2}, \tag{3.19}$$

where ℓ_1, ℓ_2 and ℓ_3 are the lengths of the sides and

$$s_\mathrm{t} = \tfrac{1}{2}(\ell_1 + \ell_2 + \ell_3), \tag{3.20}$$

it follows that A_x can be expressed in terms of ξ or x if required. Alternatively, if $x_2 - x_1$ is sufficiently small it may be good enough to evaluate A_x (or A_s) at $x = x_1$ and $x = x_2$ and use quadratic interpolation.

Once equations (3.15)–(3.17) have been integrated along the streamlines from $x = x_1$ to $x = x_2$, the pressure is known at all the streamline intercepts with the plane x_2. The pressure gradient is also known at the same points, from the three orthogonal components $\partial p/\partial s$ (known from (3.17) with $\mathrm{d}x = tx\,\mathrm{d}s$) $\partial p/\partial n$ (from (3.8)) and $\partial p/\mathrm{d}b = 0$ (from (3.9)). If we now consider the plane $x = x_2$ to contain some distorted form of figure 1 with the same lettering, the pressure gradient at E, say, may be written ∇p_E and the component in any direction $\boldsymbol{q}$ is $\boldsymbol{q}\cdot\nabla p_\mathrm{E}$. In this way we can obtain the pressure gradient along DE at both D and E, and hence the mean value, and similarly along EF to give finally a mean pressure gradient along the path DEF (which will in general not be a straight line). This must equal the pressure difference $p_\mathrm{F} - p_0$ divided by the same path length (DE + EF) thus providing one of the $2(N_\mathrm{s} - N_x - N_y)$ equations referred to in § 2. A second equation centred on the point E follows from the path BEG, and by applying the same procedure throughout the mesh (with appropriate adjustment for wall and corner points as noted in § 2) the complete set of equations is obtained. They may be written

$$f_i(v_1, \dots, v_n) = E_i = 0, \qquad i = 1, \dots, n, \tag{3.21}$$

where $n = 2(NS - NX - NY)$, $v_1, \dots, v_n$ are the unknown coefficients a_3, b_3 of equation (2.2) for each streamline (with appropriate adjustments for the wall and corner streamlines) and E_i denotes the set of errors.

The trial set v_{1i} will in general lead to a non-zero set E_i, which will clearly be changed in a nonlinear manner by changes in v_i. Newton's method has, however, been found to work well in practice, although as the solutions progress along the duct it is advisable to use quadratic extrapolation to estimate each new set of trial values v_i.

It may be expected that the governing equations will be singular as the Mach number $M \to 1$. In fact, the singularity appears in the denominator of the righthand sides of (3.16) and (3.17), where from (3.14) there follows

$$e_1 - m_{\mathrm{a}} e_2 = \frac{1}{\gamma - 1} + M^2 - \frac{pu\gamma'(T)}{\gamma m_{\mathrm{a}}(\gamma-1)^2} - \frac{m_{\mathrm{a}} u}{(\gamma-1)p} + \frac{u^2\gamma'(T)}{\gamma(\gamma-1)^2}. \tag{3.22}$$

Then, since $m_{\mathrm{a}} u/p = \gamma M^2$, (3.22) becomes

$$e_1 - m_{\mathrm{a}} e_2 = \frac{1}{\gamma - 1}\left(1 - M^2 + \frac{\gamma'(T)}{\gamma(\gamma-1)}\frac{pu}{m_{\mathrm{a}}}(\gamma M^2 - 1)\right). \tag{3.23}$$

It follows that if $\gamma'(T) = 0$, (3.16) and (3.17) are singular as $M \to 1$, but if $\gamma'(T) \neq 0$ the singularity will be shifted slightly, and since $\gamma'(T)$ is typically negative the singularity will occur at a Mach number slightly less than one. This singularity should never arise in practice in applications of the type discussed in this paper, since any approach to transonic conditions would almost certainly lead to shock formation, which has been specifically excluded.

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